Moment Invariants for 3D Flow Fields

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Abstract—Moment invariants are popular descriptors for real valued functions. Their independence from certain transformations makes them a powerful tool for the recognition of patterns and shapes. It has recently been demonstrated that the basic ideas can also be transferred to vector valued functions. Vector moment invariants can be used to define and search for interesting flow structures. A generalization to three-dimensional vector valued functions so far has not been investigated at all. In this paper, we approach that problem. We introduce a definition of moments for three-dimensional vector fields and present how flow field invariants can be constructed from the normalization of the first order vector moment tensor.

1 Introduction

The search for patterns in scalar data has a long tradition in image and shape analysis. A popular way to describe patterns are moments, which are invariant with respect to certain transformations or deformations of the patterns. They are robust, flexible, and easy to use. Moments can be interpreted as the projection of the field to a basis. Many different categories of moment invariants have been developed and analyzed [4].

The definition of features as non-local patterns is also of great interest in the area of flow analysis. For 2D flows it has already been demonstrated that moment invariants have a great potential as pattern descriptor. Thereby, two different approaches to achieve rotation invariance have been proposed. At first, there is the possibility to explicitly define a set of algebraic invariants [5] or secondly to apply the method of normalization [3], i. e. to describe the pattern with respect to a reference position [1]. In contrast to the first approach, the concept of normalization also works for higher dimensions. A generalization to three dimensions is proposed in this work.

Pattern recognition for 3D flows exhibits many new challenges compared to the two-dimensional case. This concerns the selection and the visualization of patterns but first of all the mathematical framework to provide invariant descriptors, which will be discussed in this paper. Our new definition of 3D vector moment invariants makes use of the notion of moment tensors, similar as they have been used for scalar functions in [2] and [6]. The basic idea is to arrange the moments of a given order such that they form a tensor. The characteristic directions of these tensors are then used as a reference frame to compare the moments to each other. In contrast to the scalar case, the resulting tensor for vector moments is one rank higher and not symmetric. This requires a more general approach to define a standard orientation.

Our major contribution can be summarized as:

- Provision of the theoretic framework for the definition of moments for 3D vector fields.
- Derivation of a set of flow field descriptors that are invariant with respect to rotation, background flow and velocity.
- Experiments using these descriptors for translation, rotation, and scaling invariant pattern recognition of flow fields.

2 Basics - Normalization for 3D Scalar Moments

Moments are coefficients of a function with respect to a function space basis. Usually the monomial basis $x^p y^q z^r : \Omega \to \mathbb{R}$ is used. Ordered as an array, they form a tensor. It is a contravariant tensor of rank equal to the order of the moments.

Definition 1. *For a scalar function $f : \Omega \to \mathbb{R}$, the **moment tensor** $M_{i_1 \dots i_n}$ of order $n \in \mathbb{N}$ takes the shape*

$$M_{i_1 \dots i_n} = \int_\Omega x_{i_1} \dots x_{i_n} f(x) \, \mathrm{d}^3 x. \quad (1)$$

Thereby, x_{i_j} represents the i_jth-component of x, $i_j \in \{1,2,3\}$.

Normalization is the process of putting a function into a predefined position, compare Figure 1.

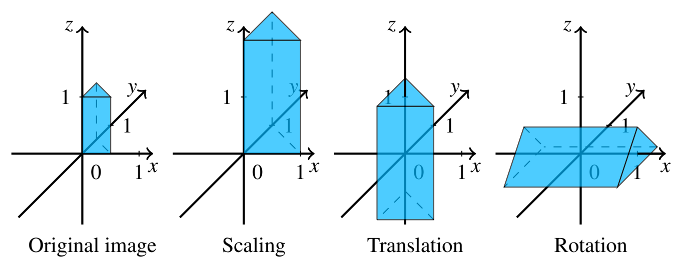

Fig. 1. Demonstration of TRS normalization for the example of the 3D characteristic function of a prism.

Normalization with respect to translation can be done by setting the first order moment tensor to zero, $M_1 = M_2 = M_3 = 0$. This is equivalent to putting the center of mass into the origin of coordinates. Normalization with respect to scaling can be achieved by demanding the moment of grade zero to be one, $M_0 = 1$. This is equivalent to the claim for unit mass. The Jordan normal form of the symmetric second order moment tensor can be used to define a standard orientation.

3 Vector Field Moments

In the following, we will demonstrate how moment invariants for 3D flow fields can be constructed from normalization of the first order vector moment tensor.

Definition 2. *For $n \in \mathbb{N}$ and a three-dimensional vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ with compact support, the n-th order* **vector moment tensor** *$M_{i_0 i_1 \dots i_n}$ is defined as*

$$M_{i_0 i_1 \dots i_n} = \int_{\mathbb{R}^3} x_{i_1} \dots x_{i_n} v_{i_0}(x) \, \mathrm{d}^3 x. \quad (2)$$

Theorem 1. *The vector moment tensor of order n is a contravariant tensor of rank $n+1$ and weight 1.*

For a general vector field, translation, rotation, and scaling can be applied to its argument and its value. That means we generally deal with six important transformations. Invariance with respect to inner translation and scaling need to be solved by searching at 'all' possible places and for 'all' possible scales in the vector field. The outer translation can be interpreted as a distortion of the pattern by some background flow or a moving frame of reference. The outer scale represents the velocity of the flow. To preserve the structure of a flow field, rotations have to be applied to both the argument and the value accordingly. In summary, the transformations of a flow field $v(x)$ with respect to which we want to normalize take the shape

$$v'(x) = sRv(R^{-1}x) + t, \tag{3}$$

with the scaling factor $s \in \mathbb{R}^+$, translational difference $t \in \mathbb{R}^3$, rotation $R \in SO(3)$. A standard position for the outer translation can be zero for the zeroth order moment tensor $M_1 = M_2 = M_3 = 0$. This erases the background flow. The velocity can be normalized by demanding a non vanishing moment to be one. Finding a standard position with respect to rotation is the hardest part. Analogously to the scalar case, we use the tensor of rank two. For vector fields, this is the first order vector moment tensor

$$\Sigma = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}. \tag{4}$$

From Theorem 1 follows that it behaves under orthogonal transformations Q like

$$\Sigma' = Q\Sigma Q^{-1}. \tag{5}$$

In contrast to the scalar case, the Jordan normal form can not be used as a standard position for this tensor. Due to its lack of symmetry, in general there is no orthogonal transformation that can bring it into this form. To overcome this problem, we make use of the Schur decomposition as standard position. It can be interpreted as a generalization of the spectral decomposition and always exists with an orthogonal transformation.

4 Results

We constructed a dataset of different flow patterns with varying positions, sizes, velocities, background flows, and orientations to give a nice overview of the behavior of the moment invariants. It is illustrated in Figure 3. The search pattern is a vortex template, i. e., a simple linear center with a Gaussian dampening as an overlay, see Figure 2 (a). For every position in the field and every scale, we render a sphere with the following properties:

- The position of the match is the center of the sphere.
- The scale of the pattern is the radius of the sphere.
- The similarity of to the vortex defines the density of the sphere.

The result can be visualized with known methods for 3D scalar fields as in Figure 4.

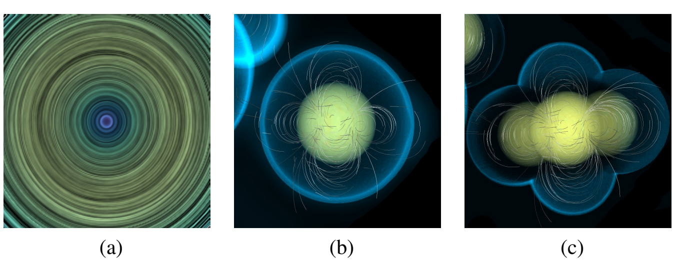

Fig. 2. (a) Search pattern. (b,c) Volume rendering of the spheres field for the quadrupole with moments computed up to grade two (b) and grade three (c) respectively.

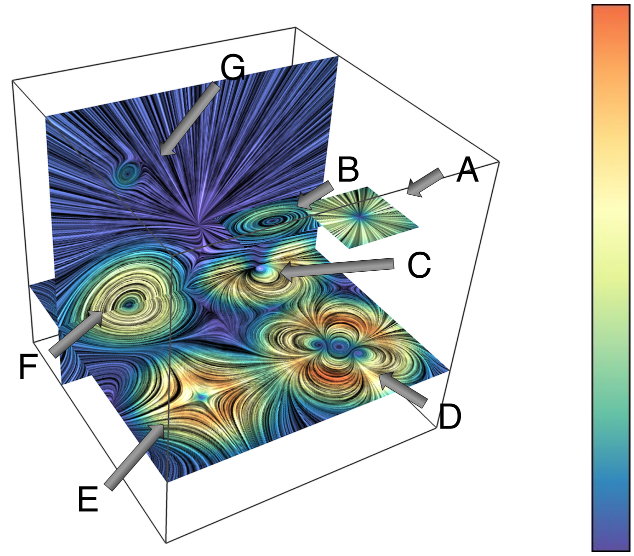

Fig. 3. LIC slides through the data set. The field contains a sink (A), an oval vortex (B), a bipole (half hidden here) (C), a vortex added to a quadrupole (D), a saddle (E), a short vortex (F), and a long vortex (G).

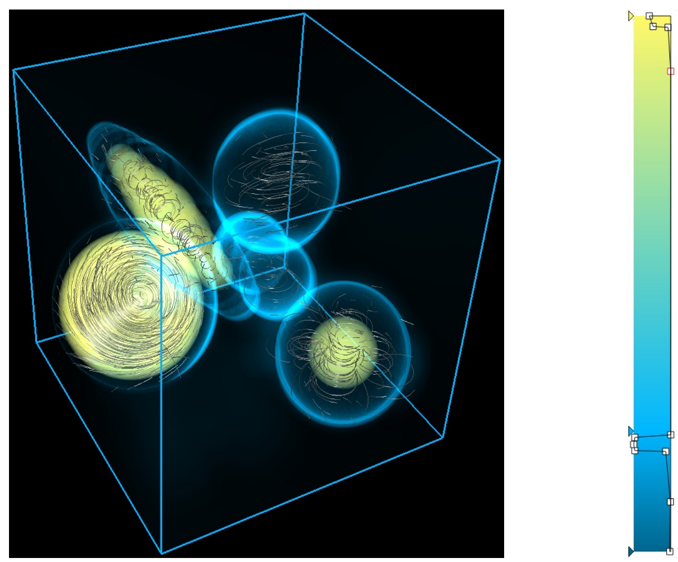

Fig. 4. Volume rendering of the spheres field with the transfer function used. The streamlines are seeded by similarity of the moments.

References

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